JOURNAL OF APPROXIMATION THEORY 26, 195–203 (1979)

An Upper Bound for the Rate of Convergence of the Hermite–Fejér Process on the Extended Chebyshev Nodes of the Second Kind

R. BOJANIC

Department of Mathematics, Ohio State University, Columbus, Ohio 43210

J. PRASAD

Department of Mathematics, California State University, Los Angeles, California 90032

AND

R. B. SAXENA

Departments of Mathematics and Astronomy, Lucknow University, Lucknow, India

Received February 22, 1977

1

The Hermite–Fejér interpolation process is an important method of approximation in the theory of approximation by interpolation. For a given family of points

$$-1 < x_{nn} < \cdots < x_{1n} < 1, \qquad n = 1, 2,...$$

and a function f on $[-1, 1]$ the Hermite–Fejér interpolatory polynomial $H_n(f, x)$ of degree $\leqslant 2n - 1$ is defined by

$$H_n(f, x_{kn}) = f(x_{kn}), H_n'(f, x_{kn}) = 0, k = 1, 2,..., n. \tag{1.1}$$

The approximation properties and the rates of convergence of the sequence $\{H_n(f, x)\}$ have been studied extensively when the points x_{kn}, $k = 1, 2,..., n$ are the zeros of the classical orthogonal polynomials.

In the simplest case, when $x_{kn} = \cos((2k - 1)\pi/2n)$, $k = 1, 2,..., n$, are the zeros of the Chebyshev polynomial $T_n(x) = \cos(n \arccos x)$, we have

$$H_n(f, x) = \sum_{k-1}^{n} f(x_{kn})(1 - xx_{kn})\left(\frac{T_n(x)}{n(x - x_{kn})}\right)^2. \tag{1.2}$$

0021-9045/79/070195–09$02.00/0

According to a well-known result of L. Fejér [8],

$$\lim_{n\to\infty} H_n(f, x) = f(x),$$

uniformly on $[-1, 1]$, for every f continuous there. A quantitative version of Fejér's result was given by R. Bojanic [5] who proved that

$$| H_n(f, x) - f(x)| \leqslant \frac{c}{n} \sum_{k=1}^{n} \omega_f\left(\frac{1}{k}\right) \tag{1.3}$$

where ω_f is the modulus of continuity of f on $[-1, 1]$ defined for every $h \geqslant 0$ by

$$\omega_f(h) = \sup\{| f(x) - f(y)| : x, y \in [-1, 1], | x - y | \leqslant h\}.$$

R. B. Saxena [10] further improved this inequality by showing that

$$| H_n(f, x) - f(x)| \leqslant \frac{c}{n} \sum_{k=1}^{n} \omega_f\left(\frac{(1 - x^2)^{1/2}}{k} + \frac{1}{k^2}\right). \tag{1.4}$$

The situation is quite different for the Hermite–Fejér interpolation process based on the zeros

$$x_{kn} = \cos \frac{k}{n+1} \pi,\ k = 1, 2,..., n,$$

of the Chebyshev polynomial of the second kind, $U_n(x) = \sin((n + 1) \operatorname{arc\,cos} x)/(1 - x^2)^{1/2}$. In this case we have

$$H_n(f, x) = \sum_{k=1}^{n} f(x_{kn})\left(1 - \frac{3x_{kn}(x - x_{kn})}{1 - x_{kn}^2}\right)(1 - x_{kn}^2)^2 \left(\frac{U_n(x)}{(n+1)(x - x_{kn})}\right)^2.$$

The sequence $\{H_n(f, x)\}$ converges to $f(x)$ pointwise on $(-1, 1)$ and uniformly on every closed subinterval of $(-1, 1)$. At the end points -1 and 1, however, the sequence diverges. We have

$$H_n(f, \pm 1) = \sum_{k=1}^{n} (1 \mp x_{kn} - 2x_{kn}^2) f(x_{kn})$$

and so

$$\begin{aligned}\lim_{n\to\infty} \frac{H_n(f, \pm 1)}{n} &= \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \left(1 \mp \cos \frac{k\pi}{n+1} - 2 \cos^2 \frac{k\pi}{n+1}\right) f\left(\frac{k\pi}{n+1}\right) \\ &= \frac{1}{\pi} \int_{-1}^{1} (1 \mp t + 2t^2) f(t) \frac{dt}{(1 - t^2)^{1/2}}\end{aligned}$$

(see [14, p. 341]).

In a case like this, it is natural to consider Egerváry–Turán's [7] modification of Hermite–Fejér's interpolation which consists in the inclusion of the end points -1 and 1 in the process of interpolation. We define a polynomial $Q_n(f, x)$ of degree $\leqslant 2n + 1$ by

$$Q_n(f, x_{kn}) = f(x_{kn}),\ Q_n(f, \pm 1) = f(\pm 1),\ Q_n'(f, x_{kn}) = 0. \tag{1.5}$$

However, in view of the relation

$$Q_n(f, x) = H_n(f, x) + (f(-1) - H_n(f, -1))\left(\frac{(1-x)\,W_n(x)}{2W_n(-1)}\right)^2 + (f(1) - H_n(f, 1))\left(\frac{(1+x)\,W_n(x)}{2W_n(1)}\right)^2, \tag{1.6}$$

where $W_n(x) = (x - x_{1n})(x - x_{2n}) \cdots (x - x_{nn})$, the Egerváry–Turán modification is interesting only if the original sequence $\{H_n(f, x)\}$ is not uniformly convergent on $[-1, 1]$. If $x_{kn} = \cos((2k-1)\pi/2n)$, $k = 1, 2,\ldots, n$, we have

$$Q_n(f, x) = \left(\frac{1+x}{2} f(1) + \frac{1-x}{2} f(-1)\right) T_n^2(x) + (1 - x^2) \sum_{k=1}^{n} f(x_{kn}) \frac{(1 + xx_{kn} - 2x_{kn}^2)}{(1 - x_{kn}^2)} \left(\frac{T_n(x)}{n(x - x_{kn})}\right)^2$$

and from (1.6) and (1.4) it follows immediately that

$$| Q_n(f, x) - f(x)| \leqslant \frac{3c}{n} \sum_{k=1}^{n} \omega_f \left(\frac{(1-x^2)^{1/2}}{k} + \frac{1}{k^2}\right).$$

A weaker form of this result was obtained earlier by D. L. Berman [4] by a different procedure. If $x_{kn} = \cos(k\pi/(n+1))$, $k = 1, 2,\ldots, n$, then

$$Q_n(f, x) = \left(\frac{1+x}{2} f(1) + \frac{1-x}{2} f(-1)\right)\left(\frac{U_n(x)}{n+1}\right)^2 + (1 - x^2) \sum_{k=1}^{n} f(x_{kn})(1 - xx_{kn})\left(\frac{U_n(x)}{(n+1)(x - x_{kn})}\right)^2$$

and the problem of uniform convergence of the sequence $\{Q_n(f, x)\}$ becomes more difficult since (1.6) cannot be used anymore. However, it was proved by P. Szász [13] that, in this case too, $\{Q_n(f, x)\}$ converges uniformly to $f(x)$ on $[-1, 1]$. His result was improved by R. B. Saxena and K. K. Mathur [11] who showed that in this case also we have the inequality

$$| Q_n(f, x) - f(x)| \leqslant \frac{c}{n} \sum_{k=1}^{n} \omega_f \left(\frac{(1-x^2)^{1/2}}{k} + \frac{1}{k^2}\right). \tag{1.8}$$

A similar result was obtained by J. Prasad and R. B. Saxena [9] for the Legendre nodes.

A further extension of the interpolation process which required that the derivative of the polynomial vanishes not only at the interior points x_{kn}, $k = 1, 2, \ldots, n$, but also at the end points -1 and 1, was studied by D. L. Berman. This time, however, the extension led to unexpected developments. Berman considered the Hermite–Fejér process of interpolation defined by

$$\begin{aligned} R_n(f, x_{kn}) &= f(x_{kn}),\ R_n(f, \pm 1) = f(\pm 1) \\ R_n'(f, x_{kn}) &= 0,\ R_n'(f, \pm 1) = 0. \end{aligned} \tag{1.9}$$

If x_{kn}, $k = 1, 2, \ldots, n$, are the zeros of the Chebyshev polynomial $T_n(x)$, then

$$\begin{aligned} R_n(f, x) = {} & f(1)(1 + (2n^2 + 1)(1 - x))\left(\frac{1 + x}{2} T_n(x)\right)^2 \\ & + f(-1)(1 + (2n^2 + 1)(1 + x))\left(\frac{1 - x}{2} T_n(x)\right)^2 \\ & + (1 - x^2)^2 \sum_{k=1}^{n} f(x_{kn}) \frac{1 + 3xx_{kn} - 4x_{kn}^2}{(1 - x_{kn}^2)^2} \left(\frac{T_n(x)}{n(x - x_{kn})}\right)^2. \end{aligned}$$

Berman proved in [1], [2] that $\{R_n(f, x)\}$ diverges at every point of $(-1, 1)$ if $f(x) = x^2$ and (with the exception of the point $x = 0$) if $f(x) = x$. Berman's results for these two special functions were generalized by R. Bojanic [6] who showed that $\{R_n(f, x)\}$ diverges on $(-1, 1)$ for every f, continuous on $[-1, 1]$, which has left and right derivatives $f_L'(1)$, $f_R'(-1)$, not both 0. More precisely, it was proved in [6] that

$$\limsup_{n\to\infty} | R_n(f, x) - f(x)| = \tfrac{3}{4}(1 - x^2)|(1 + x) f_L'(1) - (1 - x) f_R'(-1)|$$

whenever $f_L'(1)$ and $f_R'(-1)$ exist, and that, for such functions, the conditions

$$f_L'(1) = 0, f_R'(-1) = 0$$

are necessary and sufficient for the uniform convergence of the sequence $\{R_n(f, x)\}$ on $[-1, 1]$.

The situation is quite different if x_{kn}, $k = 1, 2, \ldots, n$, are the zeros of the Chebyshev polynomial $U_n(x)$. In this case we have

$$\begin{aligned} R_n(f, x) = {} & f(1)\left(1 + \left(\frac{2}{3} n^2 + \frac{4}{3} n + 1\right)(1 - x)\right)\left(\frac{(1 + x)\, U_n(x)}{2(n + 1)}\right)^2 \\ & + f(-1)\left(1 + \left(\frac{2}{3} n^2 + \frac{4}{3} n + 1\right)(1 + x)\right)\left(\frac{(1 - x)\, U_n(x)}{2(n + 1)}\right)^2 \\ & + (1 - x^2)^2 \sum_{k=1}^{n} f(x_{kn})\left(\frac{1 - xx_{kn} - 2x_{kn}^2}{1 - x_{kn}^2}\right)\left(\frac{U_n(x)}{(n + 1)(x - x_{kn})}\right)^2. \end{aligned}$$

For these polynomials $R_n(f, x)$ with f continuous on $[-1, 1]$, Berman proved in [4] that the sequence $\{R_n(f, 0)\}$ converges to $f(0)$. Subsequently, he proved in [3] that $\{R_n(f, x)\}$ converges to $f(x)$ for every $x \in [-1, 1]$ and that the convergence is uniform on each $[-1+\epsilon, 1-\epsilon]$, $0 < \epsilon < 1$. R. B. Saxena [12] improved Berman's result by showing that the convergence is uniform on $[-1, 1]$.

In the present paper we shall study the rate of convergence of the sequence $\{R_n(f, x)\}$ defined by (1.10), i.e., by (1.9) with $x_{kn} = \cos(k\pi/(n+1))$, $k = 1, 2,..., n$. Our method is based not on the explicit representation (1.10) of $R_n(f, x)$ but on the following relation between the polynomial $Q_n(f, x)$, defined by (1.5), and $R_n(f, x)$, defined by (1.9):

$$R_n(f, x) = Q_n(f, x) + (1-x)\left(\frac{(1+x)\,U_n(x)}{2(n+1)}\right)^2 Q_n'(f, 1) - (1+x)\left(\frac{(1-x)\,U_n(x)}{2(n+1)}\right)^2 Q_n'(f, -1) \tag{1.11}$$

and on the fact that for the polynomial $Q_n(f, x)$ the estimate (1.8) holds. Our main result can be stated as follows:

THEOREM 1. *Let f be a continuous function on $[-1, 1]$ and let $R_n(f, x)$ be the polynomial of Hermite–Fejér interpolation defined by* (1.9), *with $x_{kn} = \cos(k\pi/(n+1))$, $k = 1, 2,..., n$. Then for all $x \in [-1, 1]$,*

$$|R_n(f, x) - f(x)| \leqslant \frac{A}{n} \sum_{k=1}^{n} \omega_f\left(\frac{(1-x^2)^{1/2}}{k} + \frac{1}{k^2}\right) + \frac{B}{n^2}, \tag{1.12}$$

where A and B are positive constants.

If f is a non-constant function, we have

$$\begin{aligned}\frac{1}{n^2} &\leqslant \left(\frac{2}{\omega_f(1)}\right) \omega_f\left(\frac{1}{n^2}\right) \\ &\leqslant \left(\frac{2}{\omega_f(1)}\right) \frac{1}{n} \sum_{k=1}^{n} \omega_f\left(\frac{1}{k^2}\right) \\ &\leqslant \left(\frac{2}{\omega_f(1)}\right) \frac{1}{n} \sum_{k=1}^{n} \omega_f\left(\frac{(1-x^2)^{1/2}}{k} + \frac{1}{k^2}\right),\end{aligned}$$

so that,

$$|R_n(f, x) - f(x)| \leqslant \frac{c}{n} \sum_{k=1}^{n} \omega_f\left(\frac{(1-x^2)^{1/2}}{k} + \frac{1}{k^2}\right). \tag{1.13}$$

2. Some Lemmas

The proof of Theorem 1 is based on two lemmas. The first is a transformation of the basic relation (1.11).

Lemma 1. *If the polynomials* $Q_n(f, x)$ *and* $R_n(f, x)$ *are defined by* (1.5) *and* (1.9), *respectively, then*

$$R_n(f, x) = Q_n(f, x) + \frac{x(1-x^2)}{4(n+1)^2} U_n^2(x)(f(1) - f(-1))$$
$$+ \frac{n(n+2)}{6(n+1)^2}(1-x^2)(1+x) U_n^2(x)\left(f(1) - \frac{3}{n(n+2)}\sum_{k=1}^{n}\frac{f(x_{kn})}{1-x_{kn}}\right)$$
$$+ \frac{n(n+2)}{6(n+1)^2}(1-x^2)(1-x) U_n^2(x)\left(f(-1) - \frac{3}{n(n+2)}\sum_{k=1}^{n}\frac{f(x_{kn})}{1+x_{kn}}\right).$$

Proof. Differentiating (1.7) and using the formulae

$$U_n(1) = n+1,\ U_n(-1) = (-1)^n (n+1)$$

and

$$U_n'(1) = \tfrac{1}{3}n(n+1)(n+2),\ U_n'(-1) = \tfrac{1}{3}(-1)^{n-1} n(n+1)(n+2),$$

we see that

$$Q_n'(f, 1) = \left(\frac{2}{3}n^2 + \frac{4}{3}n + \frac{1}{2}\right) f(1) - \frac{1}{2} f(-1) - 2\sum_{k=1}^{n}\frac{f(x_{kn})}{1-x_{kn}}$$

and

$$Q_n'(f, -1) = \frac{1}{2} f(1) - \left(\frac{2}{3}n^2 + \frac{4}{3}n + \frac{1}{2}\right) f(-1) + 2\sum_{k=1}^{n}\frac{f(x_{kn})}{1+x_{kn}}.$$

Substitution into (1.11) yields the desired result.

Our next lemma is a quantitative version of a theorem of Berman [4] which states that for every continuous function f on $[-1, 1]$ and for $x_{kn} = \cos(k\pi/(n+1))$, $k = 1, 2, \ldots, n$,

$$\lim_{n\to\infty}\frac{1}{(n+1)^2}\sum_{k=1}^{n}\frac{f(x_{kn})}{1-x_{kn}^2} = \frac{1}{6}(f(1) + f(-1)).$$

LEMMA 2. *Let f be a continuous function on $[-1, 1]$ and let $x_{kn} = \cos(k\pi/(n+1))$, $k = 1, 2,..., n$. Then*

$$\left| f(\pm 1) - \frac{3}{n(n+2)} \sum_{k=1}^{n} \frac{f(x_{kn})}{1 \mp x_{kn}} \right| \leqslant \frac{30}{n+1} \sum_{k=1}^{n} \omega_f\left(\frac{1}{k^2}\right).$$

Proof. It is clearly sufficient to consider one choice of signs. Since

$$\sum_{k=1}^{n} \frac{1}{1 - x_{kn}} = \frac{U_n'(1)}{U_n(1)} = \frac{n(n+2)}{3},$$

we have

$$|\Delta_n(f)| = \left| f(1) - \frac{3}{n(n+2)} \sum_{k=1}^{n} \frac{f(x_{kn})}{1 - x_{kn}} \right|$$

$$= \frac{3}{n(n+3)} \left| \sum_{k=1}^{n} \frac{f(1) - f(x_{kn})}{1 - x_{kn}} \right|$$

$$\leqslant \frac{3}{n(n+2)} \sum_{k=1}^{n} \frac{|f(1) - f(x_{kn})|}{1 - x_{kn}}.$$

Hence

$$|\Delta_n(f)| \leqslant \frac{3}{n(n+2)} \sum_{k=1}^{n} \frac{\omega_f(1 - x_{kn})}{1 - x_{kn}}.$$

Since

$$1 - x_{kn} = 2 \sin^2\left(\frac{k\pi}{2(n+1)}\right),$$

we have

$$\frac{2k^2}{(n+1)^2} = \frac{2}{\pi^2}\left(\frac{k^2\pi^2}{(n+1)^2}\right) \leqslant 1 - x_{kn} \leqslant \frac{\pi^2}{2}\left(\frac{k^2}{(n+1)^2}\right) \leqslant \frac{5k^2}{(n+1)^2}.$$

Consequently

$$|\Delta_n(f)| \leqslant 15 \sum_{k=1}^{n} \frac{1}{k^2}\, \omega_f\left(\frac{k^2}{(n+1)^2}\right). \tag{2.1}$$

Next, for every $1 \leqslant k \leqslant n$,

$$\int_{k/(n+1)}^{(k+1)/(n+1)} \frac{\omega_f(t^2)}{t^2}\, dt \geqslant \omega_f\left(\frac{k^2}{(n+1)^2}\right) \int_{k/(n+1)}^{(k+1)/(n+1)} \frac{dt}{t^2}$$

$$= \frac{n+1}{k(k+1)}\, \omega_f\left(\frac{k^2}{(n+1)^2}\right)$$

$$\geqslant \frac{n+1}{2k^2}\, \omega_f\left(\frac{k^2}{(n+1)^2}\right).$$

So

$$\sum_{k=1}^{n} \frac{1}{k^2} \omega_f\left(\frac{k^2}{(n+1)^2}\right) \leqslant \frac{2}{n+1} \sum_{k=1}^{n} \int_{k/(n+1)}^{(k+1)/(n+1)} \frac{\omega_f(t^2)}{t^2}\, dt$$

$$= \frac{2}{n+1} \int_{1/(n+1)}^{1} \frac{\omega_f(t^2)}{t^2}\, dt$$

$$= \frac{2}{n+1} \int_{1}^{n+1} \omega_f\left(\frac{1}{t^2}\right) dt. \tag{2.2}$$

Using the inequality

$$\int_{k}^{k+1} \omega_f\left(\frac{1}{t^2}\right) dt \leqslant \omega_f\left(\frac{1}{k^2}\right),$$

we have

$$\int_{1}^{n+1} \omega_f\left(\frac{1}{t^2}\right) dt \leqslant \sum_{k=1}^{n} \omega_f\left(\frac{1}{k^2}\right). \tag{2.3}$$

From (2.2) and (2.3) we obtain

$$\sum_{k=1}^{n} \frac{1}{k^2} \omega_f\left(\frac{k^2}{(n+1)^2}\right) \leqslant \frac{2}{n+1} \sum_{k=1}^{n} \omega_f\left(\frac{1}{k^2}\right). \tag{2.4}$$

Finally, (2.1) and (2.4) imply

$$|\Delta_n(f)| \leqslant \frac{30}{n+1} \sum_{k=1}^{n} \omega_f\left(\frac{1}{k^2}\right).$$

3

Theorem 1 is now simple consequence of Lemmas 1 and 2. By Lemma 1,

$$\begin{aligned} &|R_n(f, x) - f(x)| \\ &\quad \leqslant |Q_n(f, x) - f(x)| + \frac{(1-x^2)\, U_n^2(x)}{(n+1)^2} (|f(1)| + |f(-1)|) \\ &\quad + (1-x^2)\, U_n^2(x) \left| f(1) - \frac{3}{n(n+2)} \sum_{k=1}^{n} \frac{f(x_{kn})}{1 - x_{kn}} \right| \\ &\quad + (1-x^2)\, U_n^2(x) \left| f(-1) - \frac{3}{n(n+2)} \sum_{k=1}^{n} \frac{f(x_{kn})}{1 + x_{kn}} \right|. \end{aligned}$$

Using (1.8), the inequality $(1 - x^2)\, U_n^2(x) \leqslant 1$, $x \in [-1, 1]$, and Lemma 2, we find that

$$|R_n(f, x) - f(x)| \leqslant \frac{c}{n} \sum_{k=1}^{n} \omega_f \left(\frac{(1 - x^2)^{1/2}}{k} + \frac{1}{k^2} \right) + \frac{|f(1)| + |f(-1)|}{(n + 1)^2}$$

$$+ \frac{60}{n + 1} \sum_{k=1}^{n} \omega_f \left(\frac{1}{k^2} \right)$$

and Theorem 1 follows.

References

1. D. L. Berman, A study of the process of Hermite–Fejér interpolation, *Dokl. Akad. Nauk SSSR* **187** (1969), 241–244 (in Russian) (*Soviet Math. Dokl.* **10** (1969), 813–816).
2. D. L. Berman, On an everywhere divergent process of Hermite–Fejér interpolation, *Izv. Vysš. Učebn. Zaved. Matematika* **92** (1970), 3–8 (in Russian).
3. D. L. Berman, A study of interpolation processes constructed on extended systems of nodes, *Izv. Vysš. Učebn. Zaved. Matematika* **105** (1971), 22–31 (in Russian).
4. D. L. Berman, Some trigonometric identities and their applications in the theory of interpolation, *Izv. Vysš. Učebn. Zaved. Matematika* **98** (1970), 26–34.
5. R. Bojanic, A note on the precision of interpolation by Hermite–Fejér polynomials, *in* "Proceedings of the Conference on Constructive Theory of Functions," pp. 69–76, Akademiai Kiadó, Budapest, 1972.
6. R. Bojanic, Necessary and sufficient conditions for the convergence of the extended Hermite–Fejér interpolation process, to appear.
7. E. Egerváry and P. Turán, Notes on interpolation *V*. On the stability of interpolation, *Acta Math. Acad. Sci. Hung.* **9** (1958), 259–267.
8. L. Fejer, Über interpolation, *Göttinger Nachr.* (1916), 66–91.
9. J. Prasad and R. B. Saxena, Degree of convergence of quasi-Hermite–Fejér interpolation, *Publ. l'Inst. Math. (Beograd)* **19** (33) (1975), 123–130.
10. R. B. Saxena, A note on the rate of convergence by Hermite–Fejér interpolation polynomials, *Canad. Math. Bull.* **17** (1974), 299–301.
11. R. B. Saxena and K. K. Mathur, The rapidity of convergence of quasi-Hermite–Fejér interpolation polynomials, *Acta Math. Acad. Sci. Hungar.* **28** (1976), 343–347.
12. R. B. Saxena, The Hermite–Fejér process on the Tchebycheff matrix of second kind, *Studia Sci. Math. Hungar.* **9** (1974), 223–232.
13. P. Szász, On quasi-Hermite-Fejér interpolation, *Acta Math. Acad. Sci. Hungar.* **10** (1959), 413–439.
14. G. Szegö, "Orthogonal polynomials," Amer. Math. Soc. Coll. Publ. XXIII, Amer. Math. Soc., New York, 1959.